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Positive $P_{0,1}$ Matrix Completion Problem for Digraphs of Order Three with Zero, One, Two and Three Arcs.

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Abstract

In this paper the positive $P_{0,1}$ -matrix completion problem is considered. It is shown that partial positive $P_{0,1}$ – matrices representing all digraphs of order 3 with $q = 0$, $q = 1$, $q = 2$ and $q = 3$ have positive $P_{0,1}$ - completion.

Keywords: Digraph; matrix completion; Partial matrix; positive $P_{0,1}$ –matrices; principal minors; sub matrix.

1. Introduction:

A **matrix** is a rectangular array of elements (entries) arranged in rows and columns, which may be added, multiplied or decomposed in various ways, making them a key concept in linear algebra and matrix theory. A matrix that has m rows and n columns is called an $m \times n$ matrix. If a matrix has n rows and n columns it is said to be a square matrix.

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A **sub matrix** of an $n \times n$ matrix A is a matrix obtained from A by deleting some rows and columns of A. For α a subset of $\{1,2,3,\dots,n\}$, a **principal sub matrix** $A(\propto)$ of A is obtained from A by deleting all rows and columns that are not in $\propto$. The determinant of such a principal sub matrix $A(\propto)$ is called the **principal minor** of $A(\propto)$ [1].

A **partial matrix** is a matrix in which some entries are specified while others are free to be chosen from a set. A **completion** of a partial matrix is the matrix obtained by making a specific choice of values for the unspecified entries so that the new matrix is of the desired type [1].

The **matrix completion problem** is concerned with determining whether or not a completion of a partial matrix exists within a certain class of matrices. As such, a description of circumstances is sought in which choices for the unspecified entries may be made from a set so that the resulting matrix, called a completion of the matrix, is of the desired type [2].

A **digraph** $G = (V, A)$ is a finite set of positive integers V, whose members are called vertices, and a set A of ordered pairs (v, u) of vertices called arcs (also called directed edges). A digraph H is said to be a **sub-digraph** of G if every vertex of H is also a vertex of G and every arc of H is also an arc of G [3].

Let G be a digraph, a path that begins and ends at the same vertex is called a **cycle**. A digraph that does not contain any cycles is called an **acyclic** digraph. A sub-digraph of a digraph is called a **clique** if it contains at least three vertices and for each pair of vertices v_i and v_j in the subset, both $v_i \to v_j$ and $v_j \to v_i$ exist [4].

A real $n \times n$ matrix is called a **positive $\boldsymbol{P_{0,1}}$ matrix** if all of its entries are positive and its principal minors are nonnegative. A partial matrix specifies a digraph if its specified entries are exactly the arcs in the digraph. In this paper, we label the specified entries as a_{ij}, the unspecified entries as x_{ij} and the diagonal entries as d_i, where the entry is in the i^{th} row and j^{th} column of the partial matrix [4].

Graphs and digraphs have been used very effectively to study matrix completion problems. Patterns that are positionally symmetric have been studied by means of their graphs while patterns that are not positionally symmetric have been studied by means of their digraphs. The class of positive $P_{0,1}$ -matrices is not symmetric and so digraphs have been used to study its completion.

Some study has been done on positive $P_{0,1}$-matrix completion. In [4], L. Hogben established that all digraphs of order two have positive $P_{0,1}$ completion. However, not much study has been done to solve this class of matrices.

The partial positive $P_{0,1}$-matrices are extracted from the digraphs as follows: A specified entry a_{ij} will be used to represent an arc in the digraph, an unspecified entry x_{ij} will be used to represent a missing arc in the digraph while d_i will specify the diagonal entries.

In many situations it is convenient to permute entries of a partial matrix. A permutation matrix Q is obtained by interchanging rows on the identity matrix. The permutation matrix P is then QPQ^T. This is represented on the digraph by renumbering the vertices. As a result of the following lemma, we can permute a partial positive $P_{0,1}$ matrix and hence renumber digraph vertices as convenient.

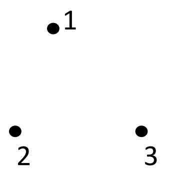
Lemma 1.1 [1]: *The class of positive $P_{0,1}$ -matrices is closed under permutation.*

Two digraphs G_1 and G_2 are said to be isomorphic if there is a one-to-one correspondence between the vertices of G_1 and those of G_2 with the property that the number of arcs joining any two vertices in G_1 is equal to the number of arcs joining the corresponding vertices in G_2. It is therefore possible to obtain G_2 from G_1 by relabeling the vertices.

In the next section all possible digraphs with 3 vertices and 0,1,2 and 3 arcs are considered and 3×3 matrices specifying these digraphs extracted. This construction of digraphs is guided by the graphs with three vertices as given in [5]. Throughout the paper, p represents the number of vertices and q represents the number of arcs of the digraph.

2. Classification of 3×3 matrices specifying digraphs with 3 vertices.

a. Consider the digraph below: $p = 3$, $q = 0$.



Let $A = \begin{bmatrix} d_1 & x_{12} & x_{13} \\ x_{21} & d_2 & x_{23} \\ x_{31} & x_{32} & d_3 \end{bmatrix}$ be the partial positive $P_{0,1}$-matrix representing the digraph. By definition of

The completion, $d_1 > 0$, $d_2 > 0$, $d_3 > 0$, considering the principal minors;

Sub matrix	Determinant
$A(1,2)$	$d_1 d_2 - x_{12} x_{21}$
$A(1,3)$	$d_1 d_3 - x_{13} x_{31}$
$A(2,3)$	$d_2 d_3 - x_{23} x_{32}$
$A(1,2,3)$	$d_1(d_2 d_3 - x_{23} x_{32}) - x_{12}(x_{21} d_3 - x_{23} x_{31}) + x_{13}(x_{21} x_{32} - d_2 x_{31})$

By definition of the completion

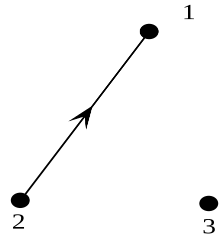
$d_1 d_2 - x_{12} x_{21} \geq 0$, $d_1 d_3 - x_{13} x_{31} \geq 0$, $d_2 d_3 - x_{23} x_{32} \geq 0$.

Choosing $x_{12} x_{21} = d_1 d_2$, $x_{13} x_{31} = d_1 d_3$, $x_{23} x_{32} = d_2 d_3$, and setting $x_{12} = \frac{d_1 d_2}{x_{21}}$, $x_{13} = \frac{d_1 d_3}{x_{31}}$, $x_{23} =$

$\frac{d_2d_3}{x_{32}}$ then $|A(1,2)| = 0$, $|A(1,3)| = 0$, $|A(2,3)| = 0$ and $|A| = \frac{d_1d_3(d_2x_{31}-x_{12}x_{32})^2}{x_{21}x_{32}x_{31}} \geq 0$.

Hence A has positive $P_{0,1}$ completion. The digraph above is a null graph and this shows that a null graph has positive $P_{0,1}$ completion.

b. Consider the digraph below: $p = 3, q = 1$



Let $B = \begin{bmatrix} d_1 & x_{12} & x_{13} \\ a_{21} & d_2 & x_{23} \\ x_{31} & x_{32} & d_3 \end{bmatrix}$ be the partial positive $P_{0,1}$-matrix representing the digraph. By definition of the completion, $d_1 > 0$, $d_2 > 0$, $d_3 > 0$ and $a_{21} > 0$. Considering the principal minors;

Sub matrix	Determinant
$B(1,2)$	$d_1d_2 - x_{12}a_{21}$
$B(1,3)$	$d_1d_3 - x_{13}x_{31}$
$B(2,3)$	$d_2d_3 - x_{23}x_{32}$
$B(1,2,3)$	$d_1(d_2d_3 - x_{23}x_{32}) - x_{12}(a_{21}d_3 - x_{23}x_{31}) + x_{13}(a_{21}x_{32} - d_2x_{31})$

By definition of the completion

$d_1d_2 - x_{12}a_{21} \geq 0$, $d_1d_3 - x_{13}x_{31} \geq 0$, $d_2d_3 - x_{23}x_{32} \geq 0$.

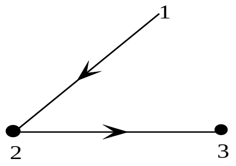
Choosing $x_{12}a_{21} = d_1d_2$, $x_{13}x_{31} = d_1d_3$, $x_{23}a_{32} = d_2d_3$, and setting $x_{13} = \frac{d_1d_3}{x_{31}}$, $x_{23} = \frac{d_2d_3}{x_{32}}$ and $x_{12} = \frac{d_1d_2}{a_{21}}$ then $|B(1,2)| = 0$, $|B(1,3)| = 0$, $|B(2,3)| = 0$ and $|B| = \frac{d_1d_3(d_2x_{31}-a_{21}x_{32})^2}{a_{21}x_{32}x_{31}} \geq 0$.

Hence B has positive $P_{0,1}$ completion.

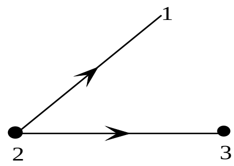
c. Consider the following digraphs: $p = 3$, $q = 2$

The digraphs in this category are:

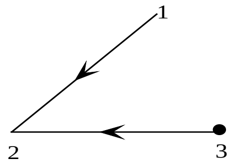
3 (i)



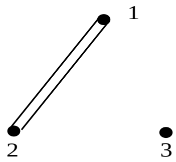
3 (ii)



3 (iii)



3 (iv)



3(i)

Let $C = \begin{bmatrix} d_1 & a_{12} & x_{13} \\ x_{21} & d_2 & a_{23} \\ x_{31} & x_{32} & d_3 \end{bmatrix}$ be the partial positive $P_{0,1}$-matrix representing the digraph 3(i) above. By definition of the completion , $d_1 > 0$, $d_2 > 0$, $d_3 > 0$, $a_{12} > 0$ and $a_{23} > 0$. Considering the principal minors;

Sub matrix	Determinant
$C(1,2)$	$d_1d_2 - a_{12}x_{21}$
$C(1,3)$	$d_1d_3 - x_{13}x_{31}$
$C(2,3)$	$d_2d_3 - a_{23}x_{32}$
$C(1,2,3)$	$d_1(d_2d_3 - a_{23}x_{32}) - x_{12}(a_{21}d_3 - x_{23}x_{31}) + x_{13}(x_{21}x_{32} - d_2x_{31})$

By definition of the completion

$d_1d_2 - a_{12}x_{21} \geq 0$, $d_1d_3 - x_{13}x_{31} \geq 0$, $d_2d_3 - a_{23}x_{32} \geq 0$.

Choosing $a_{12}x_{21} = d_1d_2$, $x_{13}x_{31} = d_1d_3$, $a_{23}x_{32} = d_2d_3$, and setting $x_{13} = \frac{d_1d_3}{x_{31}}$, $x_{21} = \frac{d_1d_2}{a_{12}}$ $x_{32} = \frac{d_2d_3}{a_{23}}$, then $|C(1,2)| = 0$, $|C(1,3)| = 0$, $|C(2,3)| = 0$ and $|C| = \frac{(d_1d_2d_3 - a_{12}a_{23}x_{31})^2}{a_{12}a_{23}x_{31}} \geq 0$.

Hence C has positive $P_{0,1}$ completion.

3(ii)

Let $D = \begin{bmatrix} d_1 & x_{12} & x_{13} \\ a_{21} & d_2 & a_{23} \\ x_{31} & x_{32} & d_3 \end{bmatrix}$ be the partial positive $P_{0,1}$-matrix representing the digraph 3(ii) above. By definition of the completion, $d_1 > 0$, $d_2 > 0$, $d_3 > 0$, $a_{21} > 0$ and $a_{23} > 0$. Considering the principal minors;

Sub matrix	Determinant
$D(1,2)$	$d_1 d_2 - x_{12} a_{21}$
$D(1,3)$	$d_1 d_3 - x_{13} x_{31}$
$D(2,3)$	$d_2 d_3 - a_{23} x_{32}$
$D(1,2,3)$	$d_1(d_2 d_3 - a_{23} x_{32}) - x_{12}(a_{21} d_3 - a_{23} x_{31}) + x_{13}(a_{21} x_{32} - d_2 x_{31})$

By definition of the completion

$d_1 d_2 - x_{12} a_{21} \geq 0$, $d_1 d_3 - x_{13} x_{31} \geq 0$, $d_2 d_3 - a_{23} x_{32} \geq 0$.

Choosing $x_{12} a_{21} = d_1 d_2$, $x_{13} x_{31} = d_1 d_3$, $a_{23} x_{32} = d_2 d_3$, and setting $x_{13} = \frac{d_1 d_3}{x_{31}}$, $x_{12} = \frac{d_1 d_2}{a_{21}}$ $x_{32} = \frac{d_2 d_3}{a_{23}}$, then $|D(1,2)| = 0$, $|D(1,3)| = 0$, $|D(2,3)| = 0$ and $|D| = \frac{d_1 d_2 (d_3 a_{21} - a_{23} x_{31})^2}{a_{21} a_{23} x_{31}} \geq 0$.

Hence D has positive $P_{0,1}$ completion.

3(iii)

Let $E = \begin{bmatrix} d_1 & a_{12} & x_{13} \\ x_{21} & d_2 & x_{23} \\ x_{31} & a_{32} & d_3 \end{bmatrix}$ be the partial positive $P_{0,1}$ -matrix representing the digraph 3(iii) above. By definition of the completion , $d_1 > 0$, $d_2 > 0$, $d_3 > 0$, $a_{12} > 0$ and $a_{32} > 0$. Considering the principal minors;

Sub matrix	Determinant
$E(1,2)$	$d_1 d_2 - a_{12} x_{21}$
$E(1,3)$	$d_1 d_3 - x_{13} x_{31}$
$E(2,3)$	$d_2 d_3 - x_{23} a_{32}$
$E(1,2,3)$	$d_1(d_2 d_3 - x_{23} a_{32}) - a_{12}(x_{21} d_3 - x_{23} x_{31}) + x_{13}(x_{21} a_{32} - d_2 x_{31})$

By definition of the completion

$d_1 d_2 - a_{12} x_{21} \geq 0$, $d_1 d_3 - x_{13} x_{31} \geq 0$, $d_2 d_3 - x_{23} a_{32} \geq 0$.

Choosing $a_{12} x_{21} = d_1 d_2$, $x_{13} x_{31} = d_1 d_3$, $x_{23} a_{32} = d_2 d_3$, and setting $x_{13} = \frac{d_1 d_3}{x_{31}}$, $x_{21} = \frac{d_1 d_2}{a_{12}}$ $x_{23} = \frac{d_2 d_3}{a_{32}}$, then $|E(1,2)| = 0$, $|E(1,3)| = 0$, $|E(2,3)| = 0$ and $|E| = \frac{d_2 d_3 (d_1 a_{32} - x_{12} x_{31})^2}{a_{32} a_{12} x_{31}} \geq 0$

Hence E has positive $P_{0,1}$ completion.

3(iv)

Let $F = \begin{bmatrix} d_1 & a_{12} & x_{13} \\ a_{21} & d_2 & x_{23} \\ x_{31} & x_{32} & d_3 \end{bmatrix}$ be the partial positive $P_{0,1}$-matrix representing the digraph 3(iv) above. By definition of the completion, $d_1 > 0$, $d_2 > 0$, $d_3 > 0$, $a_{12} > 0$ and $a_{21} > 0$. Considering the principal minors;

Sub matrix	Determinant
$F(1,2)$	$d_1d_2 - a_{12}a_{21}$
$F(1,3)$	$d_1d_3 - x_{13}x_{31}$
$F(2,3)$	$d_2d_3 - x_{23}x_{32}$
$F(1,2,3)$	$d_1(d_2d_3 - x_{23}x_{32}) - a_{12}(a_{21}d_3 - x_{23}x_{31}) + x_{13}(a_{21}x_{32} - d_2x_{31})$

By definition of the completion

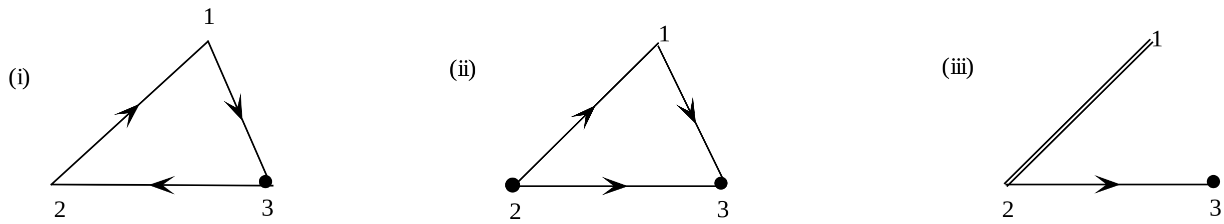
$d_1d_2 - a_{12}a_{21} \geq 0$, $d_1d_3 - x_{13}x_{31} \geq 0$, $d_2d_3 - x_{23}x_{32} \geq 0$.

Choosing $a_{12}a_{21} = d_1d_2$, $x_{13}x_{31} = d_1d_3$, $x_{23}x_{32} = d_2d_3$, and setting $x_{13} = \frac{d_1d_3}{x_{31}}$, $x_{23} = \frac{d_2d_3}{x_{32}}$ then $|F(1,2)| = 0$, $|F(1,3)| = 0$, $|F(2,3)| = 0$, and $|F| = \frac{(a_{12}d_3 - x_{32}x_{13})(d_1d_2d_3 - a_{12}x_{13}x_{32})}{x_{32}x_{31}} \geq 0$

Hence F has positive $P_{0,1}$ completion.

3. Consider the following digraphs: $p = 3$, $q = 3$

The digraphs in this category are:-



4(i)

Let $G = \begin{bmatrix} d_1 & a_{12} & a_{13} \\ a_{21} & d_2 & x_{23} \\ x_{31} & a_{32} & d_3 \end{bmatrix}$ be the partial positive $P_{0,1}$-matrix representing the digraph 4(i) above. By definition of the completion , $d_1 > 0$, $d_2 > 0$, $d_3 > 0$, $a_{12} > 0$, $a_{12} > 0$ and $a_{21} > 0$.

Considering the principal minors;

Sub matrix	Determinant
$G(1,2)$	$d_1d_2 - x_{12}a_{21}$
$G(1,3)$	$d_1d_3 - a_{13}x_{31}$
$G(2,3)$	$d_2d_3 - x_{23}a_{32}$
$G(1,2,3)$	$d_1(d_2d_3 - x_{23}a_{32}) - x_{12}(a_{21}d_3 - x_{23}x_{31}) + a_{13}(a_{21}a_{32} - d_2x_{31})$

By definition of the completion

$d_1d_2 - x_{12}a_{21} \geq 0$, $d_1d_3 - a_{13}x_{31} \geq 0$, $d_2d_3 - x_{23}a_{32} \geq 0$.

Choosing $x_{12}a_{21} = d_1d_2$, $a_{13}x_{31} = d_1d_3$, $x_{23}a_{32} = d_2d_3$, and setting $x_{31} = \frac{d_1d_3}{x_{13}}$, $x_{23} = \frac{d_2d_3}{a_{32}}$ $x_{12} = \frac{d_1d_2}{a_{21}}$, then $|G(1,2)| = 0$, $|G(1,3)| = 0$, $|G(2,3)| = 0$ and $|G| = \frac{(d_1d_2d_3 - x_{13}a_{21}a_{32})^2}{a_{21}a_{32}x_{13}} \geq 0$.

Hence G has positive $P_{0,1}$ completion.The digraph 4(i) above is cyclic and this shows that a cyclic digraph has positive $P_{0,1}$ completion.

4(ii)

Let $H = \begin{bmatrix} d_1 & x_{12} & a_{13} \\ a_{21} & d_2 & a_{23} \\ x_{31} & a_{32} & d_3 \end{bmatrix}$ be the partial positive $P_{0,1}$-matrix representing the digraph 4(ii) above. By definition of the completion , $d_1 > 0$, $d_2 > 0$, $d_3 > 0$, $a_{13} > 0$, $a_{21} > 0$ and $a_{23} > 0$. Considering the principal minors;

Sub matrix	Determinant
$H(1,2)$	$d_1d_2 - x_{12}a_{21}$
$H(1,3)$	$d_1d_3 - a_{13}x_{31}$
$H(2,3)$	$d_2d_3 - a_{23}x_{32}$
$H(1,2,3)$	$d_1(d_2d_3 - a_{23}x_{32}) - x_{12}(a_{21}d_3 - a_{23}x_{31}) + a_{13}(a_{21}x_{32} - d_2x_{31})$

By definition of the completion

$d_1d_2 - x_{12}a_{21} \geq 0$, $d_1d_3 - a_{13}x_{31} \geq 0$, $d_2d_3 - a_{23}x_{32} \geq 0$.

Choosing $x_{12}a_{21} = d_1d_2$, $a_{13}x_{31} = d_1d_3$, $a_{23}x_{32} = d_2d_3$, and setting $x_{31} = \frac{d_1d_3}{a_{13}}$, $x_{32} = \frac{d_2d_3}{a_{23}}$ $x_{12} = \frac{d_1d_2}{a_{21}}$, then $|H(1,2)| = 0$, $|H(1,3)| = 0$, $|H(2,3)| = 0$ and $|H| = \frac{d_2d_3(d_1a_{23} - a_{13}a_{21})^2}{a_{21}a_{13}a_{23}} \geq 0$

Hence H has positive $P_{0,1}$ completion. The digraph 4(ii) above is acyclic and this shows that an acyclic digraph has positive $P_{0,1}$ completion.

4(iii)

Let $K = \begin{bmatrix} d_1 & a_{12} & x_{13} \\ a_{21} & d_2 & a_{23} \\ x_{31} & x & d_3 \end{bmatrix}$ be the partial positive $P_{0,1}$-matrix representing the digraph 4(iii) above. By definition of the completion, $d_1 > 0$, $d_2 > 0$, $d_3 > 0$, $a_{12} > 0$, $a_{21} > 0$ and $a_{23} > 0$. Considering the principal minors;

Sub matrix	Determinant
$K(1,2)$	$d_1 d_2 - a_{12} a_{21}$
$K(1,3)$	$d_1 d_3 - x_{13} x_{31}$
$K(2,3)$	$d_2 d_3 - a_{23} x_{32}$
$K(1,2,3)$	$d_1(d_2 d_3 - a_{23} x_{32}) - a_{12}(a_{21} d_3 - a_{23} x_{31}) + x_{13}(a_{21} x_{32} - d_2 x_{31})$

By definition of the completion

$d_1 d_2 - a_{12} a_{21} \geq 0,\ d_1 d_3 - x_{13} x_{31} \geq 0,\ d_2 d_3 - a_{23} x_{32} \geq 0.$

Choosing $a_{12} a_{21} = d_1 d_2$, $x_{13} x_{31} = d_1 d_3$, $a_{23} x_{32} = d_2 d_3$, and setting $x_{31} = \frac{d_1 d_3}{x_{13}}$, $x_{32} = \frac{d_2 d_3}{a_{23}}$ then $|K(1,2)| = 0$, $|K(1,3)| = 0$, $|K(2,3)| = 0$, and $|K| = \frac{(a_{12} a_{23} - d_2 x_{13})(d_1 d_2 a_2 - a_{21} d_3 x_{13})}{x_{13} a_{23}} \geq 0$.

Hence K has positive $P_{0,1}$ completion.

4. Conclusion

The results of this study show that a null graph of order three has positive $P_{0,1}$ completion. It has also been shown that digraphs of order three with one or two arcs have positive $P_{0,1}$ completion. Furthermore it has been shown that order three digraphs with three arcs which are either cyclic or acyclic have positive $P_{0,1}$ completion. Study is still ongoing to try and solve this class of matrices.

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